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Structural instability of two-dimensional turbulence

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Abstract

Weakly anisotropic steady spectra are found for both inverse and direct cascades in two-dimensional turbulence of an incompressible fluid. The degree of anisotropy is shown to increase for both spectra: as $(kL)^{-2/3}$ upscales and as $(k\Lambda)^2$ downscales from the pump. A weakly anisotropic intermediate-scale pumping may thus produce a substantially anisotropic turbulence in the inertial intervals of scales.

1. Introduction

Turbulence theory is based on the assumption of local isotropy put forward by Taylor [1]. This assumption allows one to obtain universal turbulence spectra as one-parameter functions. The respective flux of the integral of motion plays the role of the spectrum parameter. Isotropy hypothesis is confirmed experimentally [2] as well as theoretically [3-5] for a small-scale turbulence produced by a quite arbitrary large-scale pump in three dimensions. Direct energy cascade is getting more isotropic as it proceeds downscales. Two-dimensional turbulence, though, admits an inverse energy cascade while a direct cascade is that of vorticity [6]. The subject of the present paper is structural stability of both direct and inverse cascades in two dimensions with respect to a weak anisotropy of the pumping.

Here we describe the properties of weakly anisotropic steady spectra of 2d turbulence and show that they may get more anisotropic while passing into the inertial intervals. We find analytically small steady anisotropic additions (neutrally stable modes) to the isotropic solutions in question. The powerful method for finding those modes goes back to Poincare. The number of such modes should be equal to the number of independent integrals of motion. Any mode in the linear approximation can be found by differentiating the general solution with respect to the value of a certain integral. The only difference of the case in question is that, instead of the integral, one should use its flux. That method was successfully applied in the theory of wave turbulence [7,8]. In order to apply it to the problem under consideration, one should find additional integrals of motion that correspond to those modes. To do this, we shall write the Euler equation in a certain symmetric form. Such a symmetry is well-hidden in terms of velocities and it is revealed in terms of canonical Clebsch variables [9,10]. This representation allows one to describe the turbulence of an incompressible fluid as a particular case of turbulence in a Hamiltonian system of quasi-particles and to use some analytical technique invented in the

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theory of wave turbulence [8]. The symmetry corresponds to the momentum of the quasi-particles that is the motion invariant whose flux is transported by the anisotropic neutrally stable modes to be found below.

The possibility to get consistent results for incompressible fluid turbulence, where no small parameter is initially at hand, arises by the following way: We assume isotropic turbulent distribution to be given and use only its general properties like scale invariance with the exponents in some interval. The analysis of anisotropic corrections can be made by using the degree of anisotropy as a small parameter. Another small parameter (the ratio of scales that interact) appears at the locality analysis. We thus can escape using uncontrollable approximations.

It is worth emphasizing that the modes are neutrally stable with respect to time. Though, being set up, those modes provide for the kind of a structural (i.e. with respect to boundary condition) instability of the spectrum if the relative anisotropy would grow while the modes expand into the respective inertial intervals. The direction of the mode propagation is governed by the sign of the momentum flux. We cannot yet find the signs of the momentum fluxes neither we can solve the initial value problem to find which of our solutions is really established. To predict the direction of an anisotropy growth, further studies (numerical and experimental) are necessary. The present state of the theory cannot predict whether the momentum flux flows downscales providing for a structural instability of the direct primary cascade (DPC) or upscales as the inverse primary cascade (IPC) or in both directions. The analysis of the present data [11] prompts that anisotropy most probably propagates upscales. If it was so, then the instability predicted allows one to describe the initial stage of the decay of initially almost isotropic turbulence, it demonstrates the onset of the self-organization process which eventually leads to a large-scale dipole vortex flow [12]. This shows also that the steady large-scale spectrum of two-dimensional turbulence may have an exponent different from $5/3$. The instability described below leads not only to the anisotropy growth but also to a somewhat steeper spectrum since the anisotropic part of the energy spectrum behaves as $k^{-7/3}$. This may be relevant to the steepening of 2d spectrum at large scales observed in numerics [13] and in the atmosphere [14] all the more that the latter data show also a substantial nonisotropy at large scales.

Another peculiarity of the problem under consideration is that odd angular harmonics of the energy spectrum are absent in the reference frame moving with a mean flow. Besides, all even harmonics in the inertial interval appear as corrections directly caused by respective angular harmonics of the damping and pumping.

The outline of the paper is as follows. We recall some basic properties of Hamiltonian approach in the Clebsch variables in Section 2.1, then we construct weak anisotropic corrections to both inverse and direct isotropic cascades at first by dimensional estimations in Section 2.2 and then by means of conformal transformations in Section 2.3. The behavior of different angular harmonics as it is dictated by the external pumping and damping is discussed in Section 2.4. Universal anisotropic energy spectra are obtained in Section 2.5. The final discussion in Section 2.6 of the question "Do anisotropic solutions set up?" closes Section 2. In Appendix A we briefly enumerate properties of the diagonal and Quasi-Lagrangian diagram techniques in Clebsch variables. This is then applied in Appendix B to restore the angular harmonics of propagators from the angular harmonics of the fourth order correlators obtaining in Section 2.2, Section 2.3. The complete convergence investigation for angular harmonics of the collision integral is given in Appendix B.

2. Clebsch variables and momentum conservation

2.1. Hamiltonian approach in the Clebsch variables

The Euler equation of motion for incompressible two-dimensional fluid could be written for the vorticity $\Omega(x,y,t)$,

$$\frac{\partial\Omega}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x}=0\,. \tag{1}$$

Here the stream function is $\psi=-\delta\mathcal{H}/\delta\omega$ where $\mathcal{H}=-\frac{1}{2}\int\psi\Omega\,dxdy$ is the energy of the fluid.

In two-dimensional geometry, it is always possible to represent the vorticity lines as an intersection of the level surfaces of two scalar fields $\lambda(x,y,t)$ and $\mu(x,y,t)$ [9]:

$$\Omega=[\nabla\lambda,\nabla\mu]\ .$$

Square brackets denote a vector product. These λ and μ are the Clebsch canonical variables, which make it possible to represent the Euler equation (1) in canonical Hamiltonian form [9,15–18]

$$\frac{\partial\lambda}{\partial t}=\frac{\delta\mathcal{H}}{\delta\mu}\,,\quad \frac{\partial\mu}{\partial t}=-\frac{\delta\mathcal{H}}{\delta\lambda}\ . \tag{2}$$

The general theory of turbulence in Hamiltonian systems is developed in terms of normal canonical variables that reveal all the conservation laws governing turbulence cascade [8]. Following [19,20] let us go over from the pair of the real variables $\lambda(\boldsymbol{x},t)$ and $\mu(\boldsymbol{x},t)$ to the complex ones $a(\boldsymbol{x},t)$ and $a^*(\boldsymbol{x},t)$,

$$\sqrt{2}a(x,y,t)=\lambda(x,y,t)+i\mu(x,y,t)\ .$$

Passing to $\boldsymbol{k}$-representation, we express the vorticity via these normal variables:

$$\Omega_k=i\int[\boldsymbol{k}_1,\boldsymbol{k}_2]a_1a_2^*\delta(\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2)\,d^2k_1d^2k_2\ .$$

Eqs. (2) have the following form in these variables:

$$i\frac{\partial a(\boldsymbol{k},t)}{\partial t}=\frac{\delta\mathcal{H}}{\delta a^*(\boldsymbol{k},t)}\,, \tag{3}$$

$$\mathcal{H}=\frac{1}{4}\int T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)a_1^*a_2^*a_3a_4\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)d^2k_1d^2k_2d^2k_3d^2k_4\,, \tag{4}$$

$$T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)=(\boldsymbol{\psi}_{13}\cdot\boldsymbol{\psi}_{24}+\boldsymbol{\psi}_{14}\cdot\boldsymbol{\psi}_{23})\,, \tag{5}$$

$$\boldsymbol{\psi}_{ij}=\boldsymbol{\psi}(\boldsymbol{k}_i,\boldsymbol{k}_j)=\frac{1}{2(2\pi)^{3/2}}(\boldsymbol{k}_i+\boldsymbol{k}_j-(\boldsymbol{k}_i-\boldsymbol{k}_j)\frac{k_i^2-k_j^2}{|\boldsymbol{k}_i-\boldsymbol{k}_j|^2})\,, \tag{6}$$

if $\boldsymbol{k}_i\neq\boldsymbol{k}_j$, otherwise $\boldsymbol{\psi}_{ij}=0$ in the reference system where a mean flow is absent. In (4) $a_j=a(\boldsymbol{k}_j,t)$. Fluid velocity in the $\boldsymbol{k}$-representation is expressed as follows:

$$v_k=-i\int\psi_{12}a_1^*a_2\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2)d^2k_1d^2k_2\ . \tag{7}$$

The velocity is a quadratic function of Clebsch variables. The main subject of interest for us is the energy spectrum which is expressed through the second correlation function of velocities and the fourth correlation function of Clebsch variables. Our main goal is to find the fourth correlation function $\langle a_1 a_2 a_3^* a_4^* \rangle = J_{1234}\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)$ that provides for a steady turbulence.

Let us mention an enormous freedom to choose different sets of Clebsch variables. By choosing independent λ and μ we actually doubled the variables since our problem should be completely determined by one scalar function (vorticity or any component of the velocity). We shall not dwell upon the ambiguity of the solution in Clebsch variables, our aim is to obtain finally the solution in terms of velocity without describing the whole class of Clebsch fields that is projected into this solution.

2.2. *Conservation law and phenomenology of turbulence spectra*

It is well known that the Euler equation (1) conserves kinetic energy $E = \int v^2 d^2x$, momentum $\boldsymbol{P} = \int \boldsymbol{v}(\boldsymbol{x},t) d^2x$ (which is zero in the reference system moving with the mean flow), and an arbitrary functional of the vorticity $\int F(\Omega)\, d^2x$, in particular, enstrophy $H = \int \Omega^2 d^2x$. According to Fjortoft's theorem, it is the flux of the energy that should determine the inverse cascade while the vorticity flux determines the small-scale part of isotropic turbulence (see e.g. [21]). To study anisotropic turbulence, one needs an integral of motion that is a vector or a tensor. Such an integral follows from the spatial homogeneity of the problem under consideration:

$$\boldsymbol{\Pi} = \int \boldsymbol{k} n(\boldsymbol{k},t) d^2k \,. \tag{8}$$

It is the total momentum of the quasi-particles with "occupation numbers"

$$n(\boldsymbol{k},t)\delta(\boldsymbol{k}-\boldsymbol{k}') \equiv \left\langle a(\boldsymbol{k},t)a^*(\boldsymbol{k}',t)\right\rangle . \tag{9}$$

Conservation of $\boldsymbol{\Pi}$ formally follows from the presence of the δ-function in (5) provided by spatial homogeneity of the problem. Note that the very possibility of the ensemble of Clebsch quasi-particles having a nonzero momentum is due to the possibility that $n(\boldsymbol{k}) \neq n(-\boldsymbol{k})$. These quasi-particles are thus similar to usual waves which could have nonzero momentum with respect to a medium that is in rest. What is important is that neither the velocity field nor the vorticity one possesses this property: because of the identity $v(\boldsymbol{k}) = v^*(-\boldsymbol{k})$, the double correlator of the velocity field is always even. Obviously, this identity follows from the fact that any flow of an incompressible fluid is completely determined by the single real quantity $\Omega(\boldsymbol{x},t)$. This means the absence of waves propagating in an incompressible fluid since any wave is described by two real variables (or one complex variable).

The existence of the additional integral of motion follows from the Kelvin theorem which is implied in the Clebsch variables. The value of the integral has no direct physical meaning since one can find different configurations of the Clebsch field having different values of the momentum $\boldsymbol{\Pi}$ but corresponding to the same velocity field. For example, changing $\boldsymbol{k} \to -\boldsymbol{k}$ for a_k one gets $\boldsymbol{\Pi} \to -\boldsymbol{\Pi}$ while the velocity field does not change. Still, the symmetry that corresponds to this integral allows us to obtain new steady anisotropic spectra that can be expressed in terms of velocities.

Knowledge of motion invariants allows one to suggest phenomenologically the expressions for the respective correlation functions. Energy is the Hamiltonian in the Clebsch variables so its density is expressed via the fourth-order correlator J_{1234}. The energy flux can thus be expressed by the usual means [20] via the simultaneous six-order correlator J_{123456}. By requiring the flux to be constant one can get the scaling exponent of $J^{(6)}$ which is $y_6 = -14$, so that $J^{(6)} \propto Pk^{-14}$ with P being the energy flux. Assuming simple scaling $y_n = An + B$, we obtain $y_n = (8n-6)/3$. The double correlator is thus $n_{\mathrm{K}}(k) \propto P^{1/3}k^{-10/3}$ which was first obtained in [19]. The fourth correlator $J_{\mathrm{K}}^{(4)} \propto P^{2/3}k^{-26/3}$ gives the energy density $E_{\mathrm{K}}(k) \propto P^{2/3}k^{-5/3}$ that can be easily recognized as the Kolmogorov 41 spectrum [22,23]. A similar procedure can be employed for the enstrophy cascade with the flux W that also can be expressed via the fourth-order correlator J_{1234}. This gives $y_n = 3n - 2$, $n_W(k) \propto W^{1/3}k^{-4}$ and $E_W(k) \propto W^{2/3}k^{-3}$. But this vorticity cascade turns out to be nonlocal and the logarithmic correction is required: $E_W(k) \propto k^{-3}\ln^{-1/3}(kL)$ [6,24].

One may argue that the way to obtain the spectra in terms of Clebsch variables is not better than the analogous speculations in terms of velocities. However, further analysis of the anisotropic parts of the spectra is much easier to do in Clebsch variables where the momentum integral is present explicitly.

If the pumping is non-isotropic then it generates momentum of the Clebsch quasi-particles. If the interaction of the quasi-particles is local in k-space (see below), then there should be the flux of momentum in the inertial intervals of scales. Since we have two inertial intervals, then the question arises which direction the momentum flux flows: downscales or upscales (or both) in each (energy and enstrophy) inertial interval. According to the revised universality concept [26], we assume the spectra in the inertial intervals to have a universal form defined by two values of the fluxes: P and $\boldsymbol{R}$ the momentum flux in the large-scale inertial interval, and W and $\boldsymbol{R}$ in the small-scale inertial interval. From the dimensional analysis, the two-flux spectra can be written as

$$n_K(\boldsymbol{k}, P, \boldsymbol{R}) = \lambda_1 P^{1/3}k^{-10/3}f_1(\xi_1) , \tag{10a}$$

$$n_W(\boldsymbol{k}, W, \boldsymbol{R}) = \lambda_2 W k^{-4} f_2(\xi_2) , \tag{10b}$$

where the dimensionless ratios of the fluxes are

$$\xi_1 = \frac{(\boldsymbol{R}\boldsymbol{k})T_{kkkk}n_k}{P}, \quad \xi_2 = \frac{(\boldsymbol{R}\boldsymbol{k})T_{kkkk}n_k k^2}{W}. \tag{11}$$

It is understandably difficult to determine the function $f(\xi)$ for all ξ (hitherto, the two-flux spectrum has been explicitly found only for acoustic turbulence [27]). We can, nevertheless, find the form of the stationary spectra in a weakly anisotropic limits when $\xi_{1,2} \ll 1$. Expanding the functions $f_1(\xi_1), f_2(\xi_2)$ at $\xi_{1,2} \ll 1$, we get a correction to the pair correlator in the form of the first angular harmonics

$$\frac{\delta n_K(\boldsymbol{k})}{n_K(k)} \propto \xi_1 \propto (kL)^{-1/3}\cos\theta_k , \tag{12a}$$

$$\frac{\delta n_W(\boldsymbol{k})}{n_W(k)} \propto \xi_2 \propto k\Lambda\cos\theta_k , \tag{12b}$$

where L is the energy containing scale for IPC and Λ is the dissipation scale for DPC. L (or Λ) dependence of the anisotropic correction (12a) (or (12b)) stems from the corresponding dependence of the momentum flux. The formula (12a) is similar to that first proposed by Kuznetsov and L'vov [29] for the three-dimensional turbulence. In the same way one can write a correction to an arbitrary simultaneous correlator: $\delta J/J \propto \xi$. For example, the fourth-order correlation functions are written as follows:

$$\delta J^{K}_{1234} \propto k_1^{-1/3}\cos\theta_1 I^{K}_{1,234} + k_2^{-1/3}\cos\theta_2 I^{K}_{2,134} + k_3^{-1/3}\cos\theta_3 I^{K*}_{3,124} + k_4^{-1/3}\cos\theta_4 I^{K*}_{4,123} , \tag{13a}$$

$$\delta J^{W}_{1234} \propto k_1\cos\theta_1 I^{W}_{1,234} + k_2\cos\theta_2 I^{W}_{2,134} + k_3\cos\theta_3 I^{W*}_{3,124} + k_4\cos\theta_4 I^{W*}_{4,123} , \tag{13b}$$

where $I^{K,W}_{(4)}$ have the same scaling properties as $J^{K,W}_{(4)}$ and are symmetric with respect to the last three arguments. In the next subsection, one of those small anisotropic corrections (13a) is shown to be an exact steady solution of the equation for the correlation functions.

The formal language we used to describe small anisotropic corrections (12) may be clarified in a more physical way without any references to the Clebsch variables. Indeed, the dimensionless ratios (12) reflect a balance between an anisotropy of the velocity shear and nonlinear mixing that tends to restore isotropy [28]:

the ratio of the characteristic time of k-eddy's approach to anisotropy due to an external shear to its (k-eddy's) turnover time should not depend on k.

Let us also note that the correction (12b) is not the only anisotropic correction to the isotropic direct cascade. Indeed, if we consider a shear flow with some anisotropic velocity profile along the z-axis $U = \int d^2r(\partial v/\partial z)r = const$ then another dimensionless ratio (instead of ξ_2) may be suggested

$$\xi_3 = \frac{UT_{kkkk}n_k k^2}{W}.$$

It gives for the corresponding anisotropic correction to the pair correlator the same scaling as the isotropic spectrum $n_W(k)$ has. Therefore, that anisotropic correction turns out to be logarithmically nonlocal. Further investigations are necessary to find suitable logarithmic dependence (probably it may be obtained by some generalization of the method used for the isotropic vorticity cascade [24]). Yet in this paper we address the spectrum (12b) with an anisotropic pumping of the momentum flux $\boldsymbol{R}$ only.

2.3. Conformal transformation and exact stationary solution

Multiplying (3) by $a^*(\boldsymbol{k},t)$ and averaging, we obtain the equation

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{St}(\boldsymbol{k},t)\,, \tag{14}$$

$$\mathrm{St}(\boldsymbol{k},t) = \mathrm{Im}\int T_{k123}J_{k123}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)d^2k_1d^2k_2d^2k_3\,, \tag{15}$$

that governs the time evolution of the pair correlator. This equation is a straightforward consequence of the Euler equation and it is the main subject of our analysis. We shall find the general (anisotropic) form of the fourth correlation function that turns rhs into zero providing for a steady turbulence.

We assume that (15) is zero for some $J = J_0$ ($J_0 = J_K$ or J_W) and consider $J = J_0 + \delta J$. To show that the rhs of (15) is zero for some $\delta J \propto \cos\theta$ (neutrally stable perturbation), we divide that integral into four identical parts, and then make in three of them the transformations that consist of the conformal dilatations invented independently by Kraichnan [30] and Zakharov [31] and rotations in $\boldsymbol{k}$-space suggested by Kats and Kontorovich [7]. For the first term this transformation $\widehat{G}_1$ looks as follows (here initial integration variables are temporarily denoted by $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}_3$ so that $\boldsymbol{k}+\boldsymbol{q}_1 = \boldsymbol{q}_2+\boldsymbol{q}_3$):

$$\widehat{G}_1: \quad \boldsymbol{q}_1 = \widehat{G}_1^2\boldsymbol{k}_1\,, \quad \boldsymbol{q}_2 = \widehat{G}_1\boldsymbol{k}_3\,, \quad \boldsymbol{q}_3 = \widehat{G}_1\boldsymbol{k}_2\,. \tag{16}$$

The operation $\widehat{G}_1$ is determined by the condition $\widehat{G}_1\boldsymbol{k}_1 = \boldsymbol{k}$ and it transforms the quadrangle $\boldsymbol{k}+\boldsymbol{k}_1 = \boldsymbol{k}_2+\boldsymbol{k}_3$ from T_{k123} into a similar quadrangle $\boldsymbol{k}+\boldsymbol{q}_1 = \boldsymbol{q}_2+\boldsymbol{q}_3$. The transformation thus relabels $\boldsymbol{k}_1$ and $\boldsymbol{k}$ and dilates all arguments of T and δJ with a factor $\lambda_1 = k/k_1$. Similar transformations

$$\boldsymbol{q}_1 = \widehat{G}_2\boldsymbol{k}_3\,, \quad \boldsymbol{q}_2 = \widehat{G}_2^2\boldsymbol{k}_2\,, \quad \boldsymbol{q}_3 = \widehat{G}_2\boldsymbol{k}_1 \quad (\widehat{G}_2\boldsymbol{k}_2 = \boldsymbol{k})\,,$$
$$\boldsymbol{q}_1 = \widehat{G}_3\boldsymbol{k}_2\,, \quad \boldsymbol{q}_2 = \widehat{G}_3\boldsymbol{k}_1\,, \quad \boldsymbol{q}_3 = \widehat{G}_3^2\boldsymbol{k}_3 \quad (\widehat{G}_3\boldsymbol{k}_3 = \boldsymbol{k})\,,$$

should be done in the second and third terms. The scattering amplitude (5) is a homogeneous function with the index 2, i.e.,

$$T(\lambda\boldsymbol{k}_1, \lambda\boldsymbol{k}_2; \lambda\boldsymbol{k}_3, \lambda\boldsymbol{k}_4) = \lambda^2 T(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{k}_3, \boldsymbol{k}_4)\,.$$

Let the correction to the fourth correlator δJ to be also a homogeneous function with its index being α. The transformation Jacobian gives λ_i^6 so in all the transformed terms acquire factors $\lambda_i^{\alpha+8}$. Therefore, after the transformations, the Eq. (15) gets the following form:

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \frac{\mathrm{Im}}{4\mathrm{k}} \int d^2k_1 d^2k_2 d^2k_3 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) T_{k1,23}(I_{k,123}+I_{1,k23}+I_{2,1k3}+I_{k,123})$$
$$\times(k^{-\alpha-8}\cos\theta_k + k_1^{-\alpha-8}\cos\theta_1 - k_2^{-\alpha-8}\cos\theta_2 - k_3^{-\alpha-8}\cos\theta_3) \; . \tag{17}$$

One can see that the rhs can be made identically equal to zero by virtue of the δ-function if $\alpha = -9$. It is exactly our case both for the inverse anisotropy cascade (13a) with $J_0 = J_{\mathrm{K}}$ and for the direct one (13b) with $J_0 = J_{\mathrm{W}}$. Note that transformations like (16) interchange zero and infinity of the integration domain in the $\boldsymbol{k}$-space. Therefore, they are possible for convergent integrals only. Using the technique introduced in [10] for expressing the asymptotics of high-order correlators via the second one, one can show (see Appendix C) that integral (15) converges with the correlator (13a) and it lies exactly on the UV convergence boundary for the correlator (13b). The perturbation (13a) is thus a steady anisotropic solution. Let us emphasize that this has been shown independently of the form of the functions $I^{\mathrm{K}}_{1,234}$ (note that we did not yet assume anisotropic part to be small; such an assumption will be used in the Appendix B to derive anisotropic corrections to the pair correlator and the Green function).

Unfortunately, one cannot make such a definite statement for the correlator (13b) due to a logarithmic UV divergence of the collision integral. It is natural to suppose that the true form of the first angular harmonic of the fourth correlation function differs from the scale-invariant expression (13b) by a logarithmic factor as well as an isotropic solution [6,24]. To get the form of the logarithmic renormalization is left for future studies, here we are interesting in power dependencies only.

2.4. *High angular harmonics*

We have found the first angular harmonic of J_{1234}. As will be shown in the next subsection, only even harmonics of J_{1234} give nonzero contribution into the velocity correlation function. Despite the fact that different angular harmonics should independently turn the rhs of (15) into zero, they are not completely independent being related by another relations imposed on the correlation functions by the Euler equation. For example, in the framework of the Wyld diagram technique, all correlation functions can be expressed in terms of propagators (see Appendix A). Assuming scale invariance, one can find the scaling exponents of the propagators that give the first harmonics (13). Then, by using these propagators one can suggest the general form of the second and subsequent harmonics of J_{1234} (see Appendix B for details). The second angular harmonics of J_{1234} are as follows for the inverse and direct cascades respectively:

$$\delta^2 J^{\mathrm{K}}_{1234} \propto L^{-2/3}[k_1^{-2/3}\cos 2\theta_1 U^{\mathrm{K}}_{1,234} + k_2^{-2/3}\cos 2\theta_2 U^{\mathrm{K}}_{2,134}$$
$$+k_3^{-2/3}\cos 2\theta_3 U^{\mathrm{K}*}_{3,124} + k_4^{-2/3}\cos 2\theta_4 U^{\mathrm{K}*}_{4,123}] \, , \tag{18a}$$
$$\delta^2 J^{\mathrm{W}}_{1234} \propto \Lambda^2[k_1^2\cos 2\theta_1 U^{\mathrm{W}}_{1,234} + k_2^2\cos 2\theta_2 U^{\mathrm{W}}_{2,134} + k_3^2\cos 2\theta_3 U^{\mathrm{W}*}_{3,124} + k_4^2\cos 2\theta_4 U^{\mathrm{W}*}_{4,123}] \; . \tag{18b}$$

Here U are some unknown functions of four arguments that have the same scaling properties as $J^{(4)}_{\mathrm{K}}$ and are symmetric with respect to the last three arguments as well as above functions I. Those expressions correspond to the squared dimensionless parameters ξ_1 and ξ_2 from (10).

Now we should study how the second angular harmonic of the collision integral (15) turns into zero. There is no such a symmetry (as for the first harmonics) that enables one to factorize (15). Still, another remarkable coincidence takes place. Let us consider, for instance, the contribution of (18a) into (15). This convergent (see Appendix C) contribution is proportional to the small parameter $(kL)^{-2/3}$ (which stems from the prefactor in (18a)). The presence of finite L (i.e. a deviation from Kolmogorov values at $k < L^{-1}$, e.g. the absence of the modes with $k < L^{-1}$) gives another contribution to (15) since J^{K} turns St into zero only at an infinite interval of k. That contribution from the distant region is small due to locality. Convergence investigation presented in Appendix C shows that for the second angular harmonic this damping contribution is proportional to exactly the same parameter $(kL)^{-2/3}$. It is thus not necessary that the second harmonic (18a) turns the collision integral into zero (it actually does not) contrary to the first harmonic (for the first harmonic, δJ is proportional to $(kL)^{-1/3}$ while the damping contribution is proportional to $(kL)^{-4/3}$). *The second angular harmonic (18a) thus appears in the inertial interval as a correction forced by a large-scale region* (in this respect, this is similar to the approach of Leslie [4] to an anisotropic 3d turbulence). That correction can be found by solving the integral equation obtained from (15):

$$\operatorname{Im} \int T_{k123} \delta^2 J^{\mathrm{K}}_{k123} \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) d^2k_1 d^2k_2 d^2k_3 + (kL)^{-2/3} c P^{2/3} k^{-8/3} \cos 2\theta_k = 0 . \tag{19}$$

As it is usual for corrections caused by a remote region [8,32], the solution of (19) has additional logarithmic factors compared to the scale-invariant form (18a): one should replace there $(k_iL)^{-2/3}$ by $(k_iL)^{-2/3} \ln^{-1}(k_iL)$. This is because of a marginal (logarithmic) infrared divergence that appears in (19) after substituting (18a).

The high ($l > 2$) angular harmonics of J_{1234} should be proportional to the $-l/3$th power of ξ:

$$\delta^l J^{\mathrm{K}} \propto (kL)^{-l/3} . \tag{20}$$

This provides for an infrared convergence of all harmonics of the collision integral (except marginal logarithmic situation for $l = 4$ – see Appendix C1 for details). The contributions of the angular harmonics with $l > 2$ into the collision integral are thus proportional to

$$\delta^l \mathrm{St}^{\mathrm{K}} \propto (kL)^{-l/3} k^{-8/3} , \tag{21}$$

and one can neglect all of them as giving corrections of the next orders with respect to the small parameter $(kL)^{-1/3}$. Still, as we go deep into the inertial interval (decreasing k) the higher the angular harmonic the faster it grows while k decreases.

The same analysis (up to unknown logarithmic factors) of the high angular corrections can be developed for the direct cascade using $(k\Lambda)$ instead of $(kL)^{-1/3}$. The results are different due to another convergence conditions. Here the Λ dependence of the high ($l > 1$) harmonics of the collision integral appears not only from the Λ dependence of J_{1234} ($\propto \Lambda^l$) but also from the divergence of the integral ($\propto \Lambda^{2-l}$). By using the results of Appendix C2 one can show that all the high ($l > 1$) angular harmonics give the same contributions in the collision integral:

$$\frac{\delta^l \mathrm{St}_k}{\mathrm{St}_k} \propto (k\Lambda)^2 . \tag{22}$$

2.5. Anisotropic energy spectra

What is accessible to experimental measurements is velocity (and its correlation functions) but not the correlation functions in the Clebsch variables. Expressing, for instance, the double velocity correlation function

$$F(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{q}) = \langle v(\boldsymbol{k})v^*(\boldsymbol{q})\rangle$$
$$= \delta(\boldsymbol{k}-\boldsymbol{q}) \int (\psi_{13}\psi_{24}) J_{12,34}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4) d^2k_1 d^2k_2 d^2k_3 d^2k_4, \tag{23}$$

one can see that odd angular harmonics of δJ_{1234} give no contribution (this follows from the identity $v(\boldsymbol{k}) = v^*(-\boldsymbol{k})$). That is why we were looking for the second angular harmonic in the preceding section. Substituting the second harmonics of the fourth-order correlation functions (18) into the expression for the double velocity correlation function (23) one obtains the power-like parts of the first nontrivial anisotropic corrections to the energy spectra

$$\frac{\delta F_{\rm K}(\boldsymbol{k})}{F_{\rm K}(k)} \propto (kL)^{-2/3}\cos(2\theta_k), \tag{24a}$$

$$\frac{\delta F_{\rm W}(\boldsymbol{k})}{F_{\rm W}(k)} \propto (k\Lambda)^{2}\cos(2\theta_k). \tag{24b}$$

As far as ln-like corrections to (24) are concerned, they appear both from $\delta^2 J_{1234}$ (see previous subsection) and from marginal divergence of integral in the rhs of (23). Let us mention, by the way, that in the IPC case both of those ln-factors are compensated.

For the inverse cascade, the even angular harmonics with the number $l > 2$ should be proportional to $L^{-l/3}$ due to the L-dependencies of $\delta^l J_{1234}$ in the integrand of (23) and the convergences of the respective integral. Therefore, in terms of the energy spectrum we see a structural instability: the higher the number of angular harmonic the higher the power of $(kL)^{-1}$ that harmonic is proportional to and the faster the respective perturbation grows as k decreases. Such a structural angular instability also takes place for a weak acoustic turbulence [33]; as a result, a substantially anisotropic yet universal spectrum appears in the inertial interval [27,8].

For the direct cascade, all even angular harmonics with the number $l > 2$ should be proportional to Λ^l due to the Λ-dependencies of $\delta^l J_{1234}$ in the integrand of (23). But those factors are partially compensated by the ultraviolet divergences Λ^{2-l} of the integral in the rhs of (23). Finally, one obtains

$$\frac{\delta F_{\rm W}(\boldsymbol{k})}{F_{\rm W}(k)} \propto (k\Lambda)^{2} f_{\rm W}(\theta_k), \tag{25}$$

where $f_{\rm W}(x)$ is some even function. Thus, *the key observation for the direct cascade is that, up to logarithmic factors, the shares of all angular harmonics of the double velocity correlator grow into the inertial interval by the same law (25).* If such corrections were set up (see below) then the steady spectrum in the inertial interval follows the angular shape of the pumping (and does not tend to a universal angular shape as for the inverse cascade).

The results of the convergence investigation of the rhs of (23), which have been used in this Subsection, can be easily obtain in the same fashion as it is done in Appendix C for the collision integral, and we do not repeat it.

2.6. *Sign of the momentum flux and the initial value problem*

We have found the steady anisotropic corrections to the spectra in the inertial intervals. To conclude whether those corrections could be set up, we should study the matching with boundary conditions i.e. with pumping and damping. Not any steady solution of a homogeneous equation (without external action) is set up under the action of quite an arbitrary pumping. To ensure that the solution found is an asymptotic state (at $t \to \infty$), one should solve the initial value problem. This could be done similarly to the weak turbulence theory [8,25], so that the criterion suggested by one of the present authors [26] is obtained: neutrally stable mode is set up in the inertial interval if it corresponds to the correct sign of the flux, i.e., it carries the flux from the pumping to the damping region. In the case of the IPC, this means that the solution (13a) should correspond to a negative momentum flux:

$$R^{\mathrm{K}} = \mathrm{Im} \int_0^k 2\pi \cos\theta_{k'} k'\, dk' \int T_{k'123} \delta^1 J^{\mathrm{K}}_{k'123} \delta(\boldsymbol{k}' + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 d^2\boldsymbol{k}_3 < 0. \tag{26a}$$

Vice versa, in the case of the DPC the solution (13b), to be set up downscales, should correspond to a positive momentum flux:

$$R^{\mathrm{W}} = \mathrm{Im} \int_0^k 2\pi \cos\theta_{k'} k'\, dk' \int T_{k'123} \delta^1 J^{\mathrm{W}}_{k'123} \delta(\boldsymbol{k}' + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 d^2\boldsymbol{k}_3 > 0. \tag{26b}$$

Thus the initial value problem is well defined, but the absence of a small parameter apparently makes a theoretical prediction yet impossible. Indeed, the present theory gives the possibility to find the scaling exponents but not the signs of the correlators (26) which are necessary to make definite predictions. We can only describe all possibilities: (i) $R^{\mathrm{K}} < 0$ – inverse cascade is structurally unstable, an anisotropic pump produces a spectrum anisotropic at large scales; (ii) $R^{\mathrm{K}} > 0$ – inverse cascade is stable, any anisotropic pumping acting in the interval of the cascade produces addition (24a) at scales smaller than those of the pump. For the direct cascade, the same is true after replacing K by W and small scales by large ones. Note that if both fluxes have "wrong" signs then both primary isotropic cascades are stable with respect to angular perturbations.

If the fluxes had right signs so that both our anisotropic corrections were formed, one can show that the process of formation obeys the same law as that of formation of isotropic spectra [25,36]: self-decelerating upscales and self-accelerating downscales (the characteristic interaction times decrease with k).

3. Conclusions

The possibility of a structural instability with respect to angular perturbations is predicted for isotropic spectra of 2d turbulence. It is shown that the weakly anisotropic steady spectra are getting more anisotropic while passing deep into the inertial intervals. To make a final conclusion on which of those anisotropic spectra could be excited by an anisotropic pump further studies are necessary.

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Appendix A. Diagrammatic technique

A.1. The diagonal diagrammatic technique in Clebsch variables

The natural scheme to develop perturbation theory for a non-equilibrium system with a strong interaction is the diagrammatic technique of the type first suggested by Wyld [34] for hydrodynamic turbulence. This technique was later generalized by Martin, Siggia and Rose [35] who demonstrated that it may be used for investigating the nonlinear dynamics of any condensed matter system. Then Zakharov and L'vov [19] extended the Wyld technique to the statistical description of hydrodynamics in the Clebsch variables. In fact this technique is also a classical limit of the Keldysh technique [37] which is applicable to any physical system described by interacting Fermi and Bose fields.

The natural objects in the Wyld diagrammatic technique are the dressed propagators, which are the Green's function $G(\boldsymbol{k},\omega)$ and correlator $n(\boldsymbol{k},\omega)$. The Green's function is defined as the average response of the Clebsch field $a(\boldsymbol{x},t)$ to a vanishingly small external "force" $f(\boldsymbol{x},t)$ which should be added to the right side of the equation of motion (3). In ω-representation

$$G(\boldsymbol{k},\omega)\delta(\boldsymbol{k}-\boldsymbol{k}')\delta(\omega-\omega') = \left\langle \frac{\delta a(\boldsymbol{k},\omega)}{\delta f(\boldsymbol{k}',\omega')} \right\rangle . \tag{A.1}$$

As a consequence of the causality principle the function $G(\boldsymbol{k},\omega)$ has to be analytic in the upper half of ω-plane. The correlator $n(\boldsymbol{k},\omega)$ is the second correlation function of the Clebsch field $a(\boldsymbol{k},t)$. In ω-representation

$$n(\boldsymbol{k},\omega)\delta(\boldsymbol{k}-\boldsymbol{k}')\delta(\omega-\omega') = \langle a(\boldsymbol{k},\omega)a^*(\boldsymbol{k}',\omega')\rangle . \tag{A.2}$$

By calling this technique diagonal we emphasize the presence of momentum δ-function in the definitions (A.1),(A.2). Using the Wyld technique one may derive the system of equations for the dressed propagators [34,19], known as the Dyson–Wyld equations:

$$G(\boldsymbol{k},\omega) = \frac{1}{\omega - \Sigma(\boldsymbol{k},\omega)}, \tag{A.3}$$

$$n(\boldsymbol{k},\omega) = |G(\boldsymbol{k},\omega)|^2 \left[\Phi_0(\boldsymbol{k},\omega) + \Phi(\boldsymbol{k},\omega)\right] . \tag{A.4}$$

Here $\Phi_0(\boldsymbol{k},\omega)$ is the correlator of the external force. The mass operators $\Sigma(\boldsymbol{k},\omega)$ and $\Phi(\boldsymbol{k},\omega)$ are the self-energy and intrinsic noise functions respectively. These are given by infinite series of one-particle irreducible terms (diagrams):

$$\begin{aligned}\Sigma &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots, \\ \Phi &= \Phi_2 + \Phi_3 + \cdots .\end{aligned} \tag{A.5}$$

In these expressions Σ_m is a functional of m vertices T, $(m-1)$ Green's functions $G(\boldsymbol{k}_j,\omega_j)$, and m correlators $n(\boldsymbol{k}_j,\omega_j)$; Φ_m is a functional of m vertices, $(m+1)$ correlators and $m-1$ Green's functions.

To reproduce the perturbation series in compact form it is convenient to use graphic notations for the objects appearing in the expansion. A wavy line represents the field a, a straight line represents $\delta/\delta f$. In accordance with (A.1),(A.2) the Green's function G has one wavy section and one straight section, while the double correlator has two wavy sections. There is only one type of vertex T, which is the junction of one straight and three wavy lines. The momentum and frequency conservation laws $\boldsymbol{k}+\boldsymbol{k}_1 = \boldsymbol{k}_2+\boldsymbol{k}_3$ and $\omega+\omega_1 = \omega_2+\omega_3$ are implied. The properties and the rules of "reading" of diagrams are given, for example, in [19]. They may be recollected by comparing the diagrams of Fig. 1 for the simultaneous fourth-order correlation function with

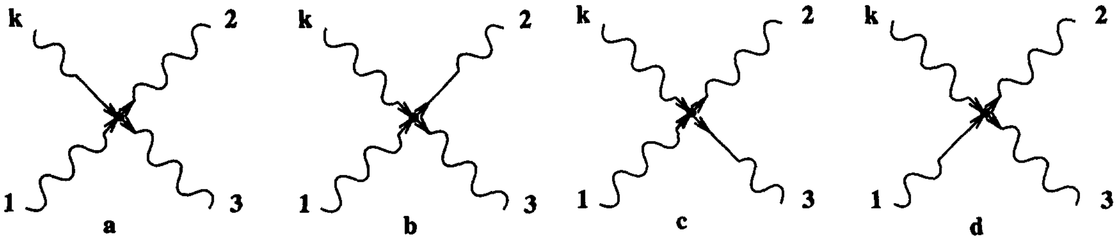

Fig. 1. Graphics representation of the perturbation series.

the following analytical expressions:

$$J_{k1,23} = \tfrac{1}{2}T(\boldsymbol{k},\boldsymbol{k}_1;\boldsymbol{k}_2,\boldsymbol{k}_3)\int d\omega d\omega_1 d\omega_2 d\omega_3 \delta(\omega+\omega_1-\omega_2-\omega_3)$$
$$\times[n_q n_1 n_2 G_3 + n_q n_1 G_2 n_3 + n_q G_1^* n_2 n_3 + G_q^* n_1 n_2 n_3]\,. \tag{A.6}$$

Here index q means $(\boldsymbol{k},\omega)$, j means $\boldsymbol{k}_j,\omega_j$ and we used shorthand notations n_j and G_j which represent $n(\boldsymbol{k}_j,\omega_i)$ and $G(\boldsymbol{k}_j,\omega_i)$.

A.2. *The quasi-Lagrangian diagrammatic technique in Clebsch variables*

The main problem limiting a use of the diagonal technique is strong power-like divergences of integrals (for example collision integral (15) both in the IPC and DPC cases. According to Kraichnan [38], divergence is caused by the strong sweeping of small eddies by large ones. A way to formally exclude such divergence is to use the quasi-Lagrangian (QL) Clebsch fields [20]. Here, we briefly enumerate some properties of QL technique without going into details which can be found in the review [20]. QL fields relate to the initial Clebsch fields by the formula

$$a(\boldsymbol{k},t) = b(\boldsymbol{r}_0,\boldsymbol{k},t_0,t)\exp\left(-i\boldsymbol{k}\int_{t_0}^{t}\boldsymbol{u}(\boldsymbol{r}_0,\boldsymbol{r}_0,t_0,\tau)d\tau\right), \tag{A.7}$$

where the QL-velocity $\boldsymbol{u}(\boldsymbol{r},\boldsymbol{r}_0,t_0,\tau)$ is expressed in the reference point $\boldsymbol{r}_0$ as follows (see (7)):

$$\boldsymbol{u}(\boldsymbol{r}_0,\boldsymbol{r}_0,t_0,\tau) = -i\int \boldsymbol{\psi}_{12} b_1^* b_2^* \exp(i(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{r}_0)d^2k_1 d^2k_2\,,$$
$$b_i = b(\boldsymbol{r}_0,\boldsymbol{k}_i,t_0,\tau)\,.$$

The equation of motion for the QL Clebsch fields may be derived by substituting (A.7) into (3). This equation differs from the Eq. (3) in the replacement of the vertex $T(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)$ by the QL vertex

$$W(\boldsymbol{r}_0;\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = (2\pi)^{-3/2}\boldsymbol{k}[\boldsymbol{\psi}_{13}\{\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)-\delta(\boldsymbol{k}-\boldsymbol{k}_2)\exp[i(\boldsymbol{k}_1-\boldsymbol{k}_3)\boldsymbol{r}_0]\}$$
$$+\boldsymbol{\psi}_{12}\{\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)-\delta(\boldsymbol{k}-\boldsymbol{k}_3)\exp[i(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{r}_0]\}]\,. \tag{A.8}$$

The dressed Green-function (A.1) and propagator (A.2) in QL technique must be rewritten in the form non-diagonal with respect to the wavevector

$$G(\boldsymbol{r}_0;\boldsymbol{k},\boldsymbol{k}',\omega)\delta(\omega-\omega') = \left\langle \frac{\delta a(\boldsymbol{r}_0;\boldsymbol{k},\omega)}{\delta f(\boldsymbol{r}_0;\boldsymbol{k}',\omega')}\right\rangle,$$

$$n(\boldsymbol{r}_0;\boldsymbol{k},\boldsymbol{k}',\omega)\delta(\omega-\omega') = \langle a(\boldsymbol{r}_0;\boldsymbol{k},\omega)a^*(\boldsymbol{r}_0;\boldsymbol{k}',\omega')\rangle . \tag{A.9}$$

Those three properties (the presence of the reference point, the new vertex and nondiagonality) lead to crucial changes in diagram technique. The graphical notations, introduced for diagonal technique, may be used nevertheless for QL technique but, of course, with another "reading" rules. The first four diagrams in QL-technique for the simultaneous fourth-order correlation function (Fig. 1) read as follows:

$$\begin{aligned} &J_{r_0;k1,23}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \\ &= \tfrac{1}{2}\int d^2k' d^2k_1' d^2k_2' d^2k_3' d\omega d\omega_1 d\omega_2 d\omega_3 \delta(\omega+\omega_1-\omega_2-\omega_3) W(\boldsymbol{r}_0;\boldsymbol{k}',\boldsymbol{k}_1';\boldsymbol{k}_2',\boldsymbol{k}_3') \\ &\times [n_{kk'}n_{11'}n_{22'}G_{33'} + n_{kk'}n_{11'}G_{22'}n_{33'} + n_{kk'}G^*_{11'}n_{22'}n_{33'} + G^*_{kk'}n_{11'}n_{22'}n_{33'}], \end{aligned} \tag{A.10}$$

where shorthand notation $n_{ii'}$ represents $n(\boldsymbol{r}_0;\boldsymbol{k}_i,\boldsymbol{k}_i',\omega_i)$. It is important to emphasize that simultaneous correlation functions in QL Clebsch fields coincide with the same functions in initial Clebsch fields and do not depend of $\boldsymbol{r}_0$. This allows us to put, without lost of generality, $\boldsymbol{r}_0 = 0$ in the rhs of the expression (A.10), as well as in all expressions for one-time correlators. Besides, it allows to connect the QL propagator (A.9) with the "occupation number" $n(\boldsymbol{k})$ (9)

$$n(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}') = \int n(\boldsymbol{r}_0;\boldsymbol{k},\boldsymbol{k}',\omega)d\omega. \tag{A.11}$$

Appendix B. Anisotropic propagators

Here, in the framework of the diagram technique in Clebsch variables (see Appendix A), we will specify, following the work [5], scaling exponents of anisotropic propagators that give the first harmonics (13).

Assuming weak anisotropy we expand the propagators (A.1,A.2) in the Fourier series

$$\begin{aligned} n(\boldsymbol{k},\omega) &= \sum_{\ell=0}^{\infty} \cos(\ell\theta)(k\lambda)^{z_\ell} n_\ell(k,\omega), \\ G(\boldsymbol{k},\omega) &= \sum_{\ell=0}^{\infty} \cos(\ell\theta)(k\lambda)^{z_\ell} G_\ell(k,\omega). \end{aligned} \tag{B.1}$$

Here λ is the energy containing scale L for inverse cascade and the dissipation scale Λ for direct cascade. z_ℓ are the scaling exponents for the dimensionless anisotropic corrections proportional to $\cos(\ell\theta)$, $z_0 = 0$.

Considering the different-time propagators $G(\boldsymbol{k},\omega)$ and $n(\boldsymbol{k},\omega)$ one necessarily finds infrared divergences of integrals even in the direct interaction approximation. The reason for this is the sweeping effect which, however, does not contribute to the results for the simultaneous correlator if the entire perturbation series is considered. The procedure of eliminating the sweeping in each order of perturbation theory is rather cumbersome [20]. Fortunately, the scaling properties of the vertex in the Lagrangian variables are the same as those of $T_{1234}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)$ so that the exponents z_ℓ can be found here without going into those complications. Let us substitute the expansions (B.1) into the Dyson–Wyld equations (A.3),(A.4). Thus we can see that z_ℓ are indeed the same for both n and G and that $n_\ell(k,\omega)$ and $G_\ell(k,\omega)$ should have the same scaling properties as $n(k,\omega)$ and $G(k,\omega)$, respectively.

Let us consider the terms with $\ell = 1$. Substituting

$$n(\boldsymbol{k},\omega) = n(k,\omega) + (k\lambda)^{z_1} n_1(k,\omega)\cos\theta_k,$$

$$G(\mathbf{k},\omega) = G(k,\omega) + (k\lambda)^{z_1} G_1(k,\omega)\cos\theta_k \tag{B.2}$$

into Eq. (A.6), one can see that in the linear approximation with respect to n_1 and G_1, the anisotropic corrections to the fourth-order correlation functions have the forms (13a) and (13b) with some analytical expression for $I_{1,234}$ if and only if $z_1 = -1/3$ and $z_1 = 1$ for inverse and direct cascade correspondingly (for the direct cascade, one drops logarithmic corrections). This is thus the justification of the formulas (12) previously obtained from dimensional analysis. Note, that corrections linear in $\cos\theta_k$ to the fourth correlator have the form (13) because of symmetry. Therefore it is clear that n_1 and G_1 do have this form in each order of perturbation theory (which also can be directly proved by the simple calculation of powers at any order and again without logarithmic corrections for the DPC solution). The only difference between considerations of the whole series and the direct interaction approximation (A.6) would be the form of analytical expressions for $I_{1,234}$. Since (13) has been shown to be a steady solution independently of the form of $I_{1,234}$, then we come to the statement, that *the anisotropic corrections (B.2) with $z_1 = -1/3(1)$ are solutions (power-like parts of solutions) of the uniform linearized equations for the double correlators in each order of perturbation theory.*

If one substitutes (B.2) into (A.4) and takes into account terms quadratic with respect to the anisotropic corrections, one obtains the exponents for the second angular harmonics: $z_2^{\mathrm{K}} = -2/3$ and $z_2^{\mathrm{W}} = 2$. Subsequent harmonics can be shown to have $z_\ell^{\mathrm{K}} = -\ell/3$ and $z_\ell^{\mathrm{W}} = \ell$ for IPC and DPC, respectively.

Substituting the first two terms of the propagator expansions (B.1) into (B.2), and restricting ourselves to the terms linear with respect to the second harmonics and quadratic with respect to the first one, we get the second angular harmonics of the fourth correlators (without logarithmic correction for the DPC case) in the form (18).

Appendix C. Convergence of the collision integral with the anisotropic corrections

To investigate the convergence of the collision integral (15) one has to examine its dependencies on large and small scales ($L \to \infty$, $\Lambda \to 0$). To do this, one can combine the QL approach in Clebsch variables (Appendix A) with the method of getting the asymptotics of correlators by sorting diagrams [10,39]. Another point that powerfully simplifies the convergence analyses is that the diagrams of different orders have the same asymptotics [20]. For the investigation of IR and UV asymptotics of (15), it is thus enough to consider only the first four diagrams of the simultaneous fourth-order correlation function (A.10) (see also Fig. 1). Let us emphasize that locality analysis employs another small parameter (the ratio of wavenumbers) in addition to a weakness of anisotropy, which makes it possible to get a consistent result without using uncontrollable approximations.

C.1. Analysis of the IR convergence

The main idea that enables one to find the asymptotics of any high-order correlators as some wavevectors go to zero is as follows: in the framework of diagram technique one can express any functions in terms of series containing pair correlators, Green functions and their small anisotropic corrections. As the scaling exponent of the pair correlator is positive, the main IR contributions into the lth harmonic of the collision integral stem from the lth harmonics of pair correlators of small wavevectors but not from the lth harmonics of Green functions. Loosely speaking, the pair correlator grows faster than the Green function as $k \to 0$. For the IPC case the exponents increase with the harmonic numbers, so the main contributions stem from the high-order harmonics of pair correlators of small wavevectors. Vice versa, for the DPC case the exponents decrease with the harmonic numbers, so the main contributions stem from the zero order harmonic of the pair correlators of small wavevectors. It means, in particular, that in the DPC case all higher harmonics of the collision integral converge logarithmically toward small k just as the zero one [6]. So, in the rest part of this Subsection one has to do only with the IPC case. The presence of the wavevector's δ-function in the integrand of St_k allows only one or two from the dummy wavevectors to be much smaller than k. Thus, to solve the problem we have to examine three cases (see Fig. 1): (a) only one from the wavevectors is small (for example $k_1 \ll k, k_2, k_3$ when only (a),(b), and (c) diagrams are essential); (b) two from the wavevectors are small, so that one of them is "going in" but another one is "going out" (for example $k_1, k_2 \ll k, k_3$ when only (a) and (c) diagrams are essential); (c)two from the wavevectors are small, so that both of them are "going out" (for example $k_3, k_2 \ll k, k_1$ when only (a) and (d) diagrams are essential).

(a) Only one from the wavevectors is small. The main contribution into the integral (A.10) over $d\boldsymbol{k}_1' d\omega_1$ stems from the region $\boldsymbol{k}_1' \simeq \boldsymbol{k}_1, \omega_1 \simeq k_1^{2/3}$. On the other hand, the main contribution into the integral (A.10) over $d\omega d\omega_2 d\omega_3$ stems from the region $\omega_2 \simeq \omega_3 \simeq \omega \simeq k^{2/3} \gg k_1^{2/3} \simeq \omega_1$. It allows us to neglect the value of ω_1 in the argument of the wavevector's δ-function in (A.10). As a result, only the function $n(\boldsymbol{k}_1, \boldsymbol{k}_1', \omega_1)$ depends on ω_1. Let us integrate the essential part of (A.10) over $d\omega_1$ with the help of $(A.11)$ and further over $\boldsymbol{k}_1'$. The expressions for T (5) and W (A.8) vertices at the leading order in the small parameter $k_1/k \ll 1$ are

$$W_{k1,23} \approx T_{k1,23}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \propto (\boldsymbol{k}_1, \boldsymbol{\psi}_{3k}+\boldsymbol{\psi}_{2k})\delta(\boldsymbol{k}-\boldsymbol{k}_2-\boldsymbol{k}_3). \tag{C.1}$$

So, we are ready to make a naive estimation of the l-st angular harmonic

$$\delta^l \mathrm{St}(\boldsymbol{k}) \propto \int_{1/L} d^2k_1 \cdots T_{k1,23} \mathrm{Im}\,\delta^l J_{k1,23} \propto \int_{1/L} d^2k_1 \cdots T_{k1,23} W_{k'1,2'3'} \delta n_l, \tag{C.2}$$

where the points mark nonessential via the integration over $\boldsymbol{k}_1$ parts of the integrand. This naive estimation is not trivial only if l is equal to 0 or 2, and vanishes due to the angle integration for all other l

$$\int_0^{2\pi} \cos(\zeta_1-\phi_1)\cos(\zeta_1-\phi_2)\cos(l\zeta_1)d\zeta_1 = 0, \tag{C.3}$$

where ζ_1 is the angle between $\boldsymbol{k}_1$ and the momentum flux $\boldsymbol{R}$, ϕ_1 is the angle between $\boldsymbol{R}$ and the vector $\boldsymbol{\psi}_{3k}+\boldsymbol{\psi}_{2k}$, ϕ_2 is the angle between $\boldsymbol{R}$ and the vector $\boldsymbol{\psi}_{3'k'}+\boldsymbol{\psi}_{2'k'}$. The second harmonic stands out from all the others since the asymptotics of two vertices, T and W, contain cosines. To test convergences of another harmonics of St_k we must expand the integrand in the series with respect to the small wavevector and frequency. The ω-expansion does not give us a nontrivial result because no new dependencies on the angle ζ_1 appear. As far as the first angular harmonic of the collision integral is concerned, already the first order correction over $\boldsymbol{k}_1$ gives nontrivial result. To estimate a convergence of some higher ($l>2$) harmonic let us note that the integral $\int_0^{2\pi} f(\zeta_1)\cos(l\zeta_1)d\zeta_1$ is not trivial only if the lth harmonic in the angular expansion of $f(\zeta_1)$ exists (here some new angular dependence of the integrand on the angle ζ_1, appeared from the $\boldsymbol{k}_1$ expansion, suppresses in some function f). But those nontrivial terms come for the first time only from the $(l-2)$ order of the integrand's expansion over $\boldsymbol{k}_1$. So, the angular harmonics of the collision integral converge at $L \to \infty$ as

$$\delta^l \mathrm{St} \propto \begin{cases} L^{-4/3} & \text{if } l=1\,, \\ L^{-\frac{2}{3}(l-2)} & \text{if } l>1\,. \end{cases} \tag{C.4}$$

In particular, one obtain that the second angular harmonic of the collision integral converges logarithmically at small k.

We studied here only the case, when a small wavevector is "going in", but it is easy to see, that the case when a small wavevector is "going out" gives the same result.

(b) Two from the wavevectors are small, so that one of them is "going in" but another one is "going out". $k_1, k_2 \ll k, k_3$, so only (a) and (c) diagrams are essential. By the same way as it was done in the previous subsection, one can fulfill integration over $\omega_{1,2,3}$ and $\boldsymbol{k}'_{1,2,3}$. The vertices in the leading order in the small parameters $k_1/k, k_2/k \ll 1$ are

$$W_{k1,23} \propto \left[(\boldsymbol{k}_1, \boldsymbol{k}_2) - \frac{(\boldsymbol{k}_1, \boldsymbol{k})(\boldsymbol{k}_2, \boldsymbol{k})}{k^2}\right] \delta(\boldsymbol{k} - \boldsymbol{k}_3)\,, \tag{C.5}$$

$$T_{k1,23} \propto (\boldsymbol{k}, \boldsymbol{\psi}_{12})\,. \tag{C.6}$$

The first contribution from the lth harmonic of the four-point correlator $\delta^l J_{k1,23}$ to $\delta^l \mathrm{St}$,

$$\begin{aligned} \mathrm{Im}\delta^l J_{k1,23} \approx \int \delta^l(n_1 n_2)((\boldsymbol{k}_1, \boldsymbol{k}_2) - \frac{(\boldsymbol{k}_1, \boldsymbol{k}')(\boldsymbol{k}_2, \boldsymbol{k}')}{k'^2}) \\ \times [n(\boldsymbol{k}, \boldsymbol{k}'; \omega)\mathrm{Im}\, G(\boldsymbol{k}, \boldsymbol{k}'; \omega) + \mathrm{Im}\, G^*(\boldsymbol{k}, \boldsymbol{k}'; \omega) n(\boldsymbol{k}, \boldsymbol{k}')]\, d^2k' d\omega\,, \end{aligned} \tag{C.7}$$

is equal to zero due to the cancellation of two terms in the square brackets. So, one has to take into account terms of the next order in small wavevectors and frequencies. Since $\omega_{1,2} \ll k_{1,2}$, one obtains for $\delta^l \mathrm{St}$

$$\begin{aligned} \delta^l \mathrm{St}(\boldsymbol{k}) \propto \int \delta(n(\boldsymbol{k}_1, \omega_1) n(\boldsymbol{k}_2, \omega_2)) \left((\boldsymbol{k}_1, \boldsymbol{k}_2) - \frac{(\boldsymbol{k}_1, \boldsymbol{k}')(\boldsymbol{k}_2, \boldsymbol{k}')}{k'^2}\right) (\boldsymbol{k}, \boldsymbol{\psi}_{12})(\omega_1 - \omega_2) \\ \times [n(\boldsymbol{k}, \boldsymbol{k}'; \omega) \frac{\partial}{\partial \omega} \mathrm{Im}\, G(\boldsymbol{k}, \boldsymbol{k}'; \omega) + \mathrm{Im}\, G^*(\boldsymbol{k}, \boldsymbol{k}'; \omega) \frac{\partial}{\partial \omega} n(\boldsymbol{k}, \boldsymbol{k}'; \omega)]\, d^2k_1 d^2k_2 d^2k' d\omega d\omega_1 d\omega_2\,. \end{aligned} \tag{C.8}$$

This term again is equal to zero, now due to antisymmetry of the integrand with respect to the interchange: $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2, \omega_1 \leftrightarrow \omega_2$. Moreover, one sees that nontrivial contribution to $\delta \mathrm{St}$ can arrive only from a calculation of the vertex asymmetry. The calculation of the first asymmetric correction to the W vertex gives a nontrivial result for the second and fourth harmonics of the collision integral and zero for another ones due to the angle integrations. If one eliminates the cancellations in (C.7) by expanding the integrand over small wavevectors (not over small frequencies as it was done in (C.8)) and takes into account the first asymmetric correction to the vertices one obtain nontrivial results for the first and third harmonics. Following the above line of arguments (see the previous a-case), we conclude that the angle integrations in some high lth harmonic ($l > 4$) of the collision integral may be nontrivial only for terms in the integrand containing a large enough (greater than l) power of $k_{1,2}$. Thus, we get the estimations

$$\delta^l \mathrm{St} \propto \begin{cases} L^{l/3-7/3} & \text{if } l = 1, 3\,, \\ L^{l/3-2} & \text{if } l = 2, 4\,, \\ L^{-2l/3+8/3} & \text{if } l > 4\,, \end{cases} \tag{C.9}$$

that show the power-like convergences of all harmonics of the collision integral.

(c) Two from the wavevectors are small and both of them are "going out". $k_3, k_2 \ll k, k_1$ so only (a) and (d) diagrams are essential. As in the case (a), one can fulfill integration over $\omega_{1,2,3}$ and $\boldsymbol{k}'_{1,2,3}$. The vertices in the leading order of the small parameters $k_2/k, k_3/k \ll 1$ are as follows:

$$W_{k1,23} \approx T_{k1,23} \delta(\boldsymbol{k} + \boldsymbol{k}_1) \propto (\boldsymbol{k}_2 \boldsymbol{k}_3) \delta(\boldsymbol{k} + \boldsymbol{k}_1)\,. \tag{C.10}$$

The naive estimation of the lth angular harmonic of the collision integral

$$\delta^l\,\mathrm{St} \propto \int_{1/L} \delta^l(n_3 n_2)(\boldsymbol{k}_2\boldsymbol{k}_3)^2 d^2k_2 d^2k_3 \tag{C.11}$$

gives a nontrivial result for $l = 0,4$ and zero due to the angle integrations for all other l. To find the leading L dependences for $l \neq 0,4$ we must expand the integrand up to the next terms in k_1, k_2. One thus obtains

$$\delta^l\mathrm{St} \propto \begin{cases} L^{l/3-7/3} & \text{if } l = 1,3\,, \\ L^{-8/3} & \text{if } l = 2\,, \\ L^{-2l/3+8/3} & \text{if } l > 3\,. \end{cases} \tag{C.12}$$

The angular harmonics of the collision integral converge logarithmically for $l = 4$ and power-like for all other l.

C.2. Analysis of the UV convergence

Since the scaling exponent of the pair correlator is positive, then the main UV contributions into the lth harmonic of the collision integral stem from the lth harmonics of the Green functions of large wavevectors. For the direct cascade, the exponent decreases with the harmonic number so that the main contributions stem from the high-order harmonics of propagators of large wavevectors. For the inverse cascade, the exponents increase with the harmonic numbers so that the main contributions stem from the zero order harmonics of propagators of large wavevectors. This means, in particular, that in the case of IPC all the harmonics of the collision integral have the same convergence reserve (interval of exponents providing convergence) as the zero one [20]. Therefore, in the rest part of this Subsection we have to do only with the DPC case. Now, only two or three wavevectors in J_{1234} may be large. We thus have to examine three cases similarly to the IR case.

(a) Two from the wavevectors are large, so that one of them is "going in" but another one is "going out". $k_1, k_2 \gg k, k_3$ and only (c) and (d) diagrams are essential. Due to the presence of small parameters $k/k_1, k/k_2 \ll 1$ and also due to the smallness of the respective frequencies (that was proved to be true for the DPC case [24]) one can fulfill integration over $\omega, \omega_2, \omega_3$ and $\boldsymbol{k}', \boldsymbol{k}_2', \boldsymbol{k}_3'$. The asymptotics of vertices are as follows:

$$W_{k1,23} \propto \left((\boldsymbol{k},\boldsymbol{k}_3) - \frac{((\boldsymbol{k}_1,\boldsymbol{k})(\boldsymbol{k}_1,\boldsymbol{k}_3)}{k_1^2} \right) \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\,, \tag{C.13}$$

$$T_{k1,23} \propto (\boldsymbol{k}_1, \boldsymbol{\psi}_{k3})\,. \tag{C.14}$$

The leading contribution from the lth angular harmonic of the fourth correlator, $\delta^l J_{k1,23}$ to $\delta^l\mathrm{St}$

$$\begin{aligned} \mathrm{Im}\delta^l J_{k1,23} \propto \int^{1/\Lambda} & \delta^l[n(\boldsymbol{k}_1,\boldsymbol{k}_1';\omega_1)\mathrm{Im}G(\boldsymbol{k}_1,\boldsymbol{k}_1';\omega_1) + \mathrm{Im}G^*(\boldsymbol{k}_1,\boldsymbol{k}_1';\omega_1)n(\boldsymbol{k}_1,\boldsymbol{k}_1')] \\ & \times n_k n_3 \left((\boldsymbol{k},\boldsymbol{k}_3) - \frac{(\boldsymbol{k}_1',\boldsymbol{k})(\boldsymbol{k}_1',\boldsymbol{k}_3)}{k_1'^2} \right) d^2\boldsymbol{k}'_1 d\omega_1\,, \end{aligned} \tag{C.15}$$

is equal to zero due to the cancellation of two terms in the square brackets. So, one has to take into account terms of the next order in the small wavevectors and frequencies. Since $\omega_{k,3} \ll k, k_3$, one obtains

$$\delta^l\mathrm{St} \propto \int^{1/\Lambda} n(\boldsymbol{k},\omega)n(\boldsymbol{k}_3,\omega_3)(\omega - \omega_3) \left((\boldsymbol{k},\boldsymbol{k}_3) - \frac{(\boldsymbol{k},\boldsymbol{k}_1')(\boldsymbol{k}_3,\boldsymbol{k}'_1)}{k_1'^2} \right) (\boldsymbol{k}_1, \boldsymbol{\psi}_{k3})$$

$$\times\delta^l\left[n(\boldsymbol{k}_1,\boldsymbol{k}'_1;\omega_1)\frac{\partial}{\partial\omega_1}\mathrm{Im}G(\boldsymbol{k}_1,\boldsymbol{k}'_1;\omega_1)+\mathrm{Im}G^*(\boldsymbol{k}_1,\boldsymbol{k}'_1;\omega_1)\frac{\partial}{\partial\omega_1}n(\boldsymbol{k}_1,\boldsymbol{k}'_1)\right]$$
$$\times d^2k_1d^2k_3d^2k'_1d\omega d\omega_1d\omega_3\,. \tag{C.16}$$

This is zero after integration over ω,ω_3. Expanding the expression in the square brackets at the rhs of (C.15) in the small wavevectors, one obtains a nontrivial result for all angular harmonics,

$$\delta^l\mathrm{St}\propto\Lambda^{2-l}\,. \tag{C.17}$$

This shows the power-like divergence for $l>2$, convergence for $l=1$ and marginal (logarithmic) divergence for $l=2$.

(b) Two from the wavevectors are large, so that both of them are "going out". $k_2,k_3\gg k,k_1$ and only (b) and (c) diagrams are essential. In the leading order in the small parameters $k/k_{2,3}\ll 1$ one can fulfill the integration over $\omega,\omega_{1,3}$ and $\boldsymbol{k}',\boldsymbol{k}'_{1,3}$. The vertices are $W_{k1,23}\approx T_{k1,23}\delta(\boldsymbol{k}_2+\boldsymbol{k}_3)\propto(\boldsymbol{k}\boldsymbol{k}_1)\delta(\boldsymbol{k}_2+\boldsymbol{k}_3)$.

Already the naive estimation of the lth angular harmonic of the collision integral gives a nontrivial result

$$\delta^l\mathrm{St}(\boldsymbol{k})\propto\int^{1/\Lambda}\ldots\delta^l(\mathrm{Im}G(\boldsymbol{k}_2,\boldsymbol{k}'_2,\omega_2)n(-\boldsymbol{k}_2,-\boldsymbol{k}'_2,-\omega_2)$$
$$+\mathrm{Im}G(-\boldsymbol{k}_2,-\boldsymbol{k}'_2,-\omega_2)n(\boldsymbol{k}_2,\boldsymbol{k}'_2,\omega_2))d^2k_2d^2k'_2d\omega_2\propto\Lambda^{2-l}, \tag{C.18}$$

which coincides with the previous case (C.17).

(c) Three from the wavevectors are large $k_{1,2,3}\gg k$ and only (b), (c) and (d) diagrams are essential. Here one integrates the leading order in the small parameters $k/k_{1,2,3}\ll 1$ over ω and $\boldsymbol{k}'$. The vertices are $W_{k1,23}\approx T_{k1,23}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\propto(\boldsymbol{k},\boldsymbol{\psi}_{31}+\boldsymbol{\psi}_{21})\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$.

The naive estimation of the lth angular harmonic of the collision integral gives the same nontrivial result as in both a- and b-cases

$$\delta^l\mathrm{St}\propto\int^{1/\Lambda}(\boldsymbol{k},\boldsymbol{\psi}_{31}+\boldsymbol{\psi}_{21})(\boldsymbol{k},\boldsymbol{\psi}_{3'1'}+\boldsymbol{\psi}_{2'1'})\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\delta(\boldsymbol{k}'_1-\boldsymbol{k}'_2-\boldsymbol{k}'_3)$$
$$\times\delta^l[n(\boldsymbol{k}_1,\boldsymbol{k}'_1,\omega_1)\mathrm{Im}G(\boldsymbol{k}_2,\boldsymbol{k}'_2,\omega_2)n(\boldsymbol{k}_3,\boldsymbol{k}'_3,\omega_3)+n(\boldsymbol{k}_1,\boldsymbol{k}'_1,\omega_1)n(\boldsymbol{k}_2,\boldsymbol{k}'_2,\omega_2)\mathrm{Im}G(\boldsymbol{k}_3,\boldsymbol{k}'_3,\omega_3)$$
$$+\mathrm{Im}G(\boldsymbol{k}_1,\boldsymbol{k}'_1,\omega_1)n(\boldsymbol{k}_2,\boldsymbol{k}'_2,\omega_2)n(\boldsymbol{k}_3,\boldsymbol{k}'_3,\omega_3)]d^2k_1d^2k_2d^2k_3d^2k'_1d^2k'_2d^2k'_3d\omega d\omega_1d\omega_2d\omega_3$$
$$\propto\Lambda^{2-l}. \tag{C.19}$$

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